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A METHOD OF DETERMINING THE EXTRANEOUS UNKNOWNS IN PROBLEMS OF THE STABILITY AND VIBRATIONS OF RODS†

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An approximate method of determining the critical loads in problems of the stability of compressed rods has been extended to statically indeterminate systems. For this purpose, a method has been developed for solving stability problems when there are extraneous unknowns defined by the stationarity condition for the potential energy of the system. It is shown that, combined with Grammel's method and Hamilton's variational principle, the method described for determining the extraneous unknowns in statically indeterminate systems can also be used in problems of finding the natural frequencies of vibrations of rods. © 2002 Elsevier Science Ltd. All rights reserved.

1. THE ENERGY METHOD IN STABILITY THEORY

In problems of the stability of rods compressed within the limits of elasticity, the energy method is widely used [1–5]. In this method, when there is loss of stability without elongation of the axis, the critical force is given approximately by means of the formula

$$P_* = \frac{U}{\lambda} = \int_0^l EJy''^2 dz \left(\int_0^{l_0} y'^2 dz \right)^{-1} \tag{1.1}$$

where U is the strain potential energy, λ is the displacement of the point of application of the longitudinal force, EJ is the minimum stiffness, l is the rod length, and $y(z)$ is the approximate dependence of the deflection on the longitudinal coordinate. Here, the strain potential energy is usually defined by the expression

$$U = \frac{1}{2}\int_0^l EJy''^2 dz \tag{1.2}$$

At the same time, the strain energy can be calculated [1] directly in the form

$$U = \frac{1}{2}\int_0^l \frac{M^2 dz}{EJ} \tag{1.3}$$

where M is a function of the bending moment, defined by the condition of equilibrium of the intercepted part of the rod in the strained state.

It is well known [1, 3] that the use of expression (1.3) is preferable to (1.2) since, when calculating U by means of formula (1.3), the accuracy of the approximate solution depends on the accuracy of the definition of $y(z)$ and not on the accuracy of the approximation of $y''(z)$, which in a number of cases can turn out to be lower than the accuracy of the selection of $y(z)$.

Furthermore, the energy method, based on the use of expression (1.2), leads to an inaccurate result in the case of rods of piecewise-constant stiffness. To confirm the above, consider the example given by V. L. Biderman.‡

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‡BIDERMAN, V. L., The energy method in the theory of the stability of compressed rods. Paper read at the *Departmental Conference on Applied Mechanics (Resistance of Materials and Dynamics and Strength of Machines)*, N. E. Bauman Moscow State Technological University, 16 April 1993.

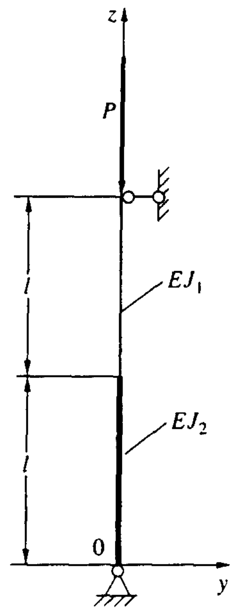

Fig. 1

Figure 1 shows a statically determinate rod compressed along its axis by a force P. To determine the critical force, we specify a function of the deflections in the form

$$y = A\sin(\pi z/(2l))$$

Then, by formula (1.1)

$$P_* = \frac{\pi^2}{8l^2}(EJ_1 + EJ_2) \tag{1.4}$$

i.e. when $EJ_1 \to 0$ we have $P_* \neq 0$, although this should be $P_* \to 0$, since the upper section supports no load.

Solving the same problem using expression (1.3), we obtain $M = -Py$, and consequently

$$U = \frac{1}{2}\int_0^{2l}\frac{M^2 dz}{EJ} = \frac{P^2A^2l}{4}\left(\frac{1}{EJ_1} + \frac{1}{EJ_2}\right)$$

Therefore, on the basis of the energy equation [2]

$$U = P_*\lambda \tag{1.5}$$

we have

$$P_* = \frac{\pi^2}{2l^2}\left(\frac{1}{EJ_1} + \frac{1}{EJ_2}\right)^{-1} \tag{1.6}$$

In this case, the shortcoming noted above is not present, since, when $EJ_1 \to 0$ or $EJ_2 \to 0$, $P_* \to 0$, which is consistent with the physical sense of the problem.

In particular, we will compare the accuracy of the solution of the problem for the example in question when $EJ_2 = 4EJ_1 = 4EJ$. We have

$$P_* = \xi\frac{EJ}{l^2}, \quad \xi = \begin{cases} 5\pi^2/8 = 6.16 & \text{by formula (1.4)} \\ 2\pi^2/5 = 3.94 & \text{by formula (1.6)} \\ 3.65 & \text{for the accurate solution [2].} \end{cases}$$

It can be seen that, when there is considerable inhomogeneity of the cross-section, the energy method based on a calculation of the potential energy using expression (1.3) leads to considerably better agreement with the accurate solution than the method based on the use of expression (1.2).

Both approaches are based on an approximate specification of the function of the deflections $y(z)$, i.e. the same field of displacements. Therefore, both methods give an overestimate of the critical force. The fundamental difference between these two approaches lies in the method for calculating U.

Further development of the method using the expression $M = M(P, y, z)$ involves its extension to statically indeterminate problems, a feature of which is the existence of extraneous unknowns, not determined by the equations of statics.

2. THE GENERAL APPROACH

The general approach to solving statically indeterminate problems of the stability of a rod compressed by a force P reduces to the following.

1. Taking into account the boundary conditions of the problem, the deflection function $y = Af(z)$ is specified approximately, where A is an undefined coefficient.
2. Using expression (1.3) we determine the strain potential energy $U = U(X_1, X_2, \dots, P, A)$, where $X_1, X_2, \dots$ are extraneous unknowns of the statically indeterminate problem.
3. We set up an expression for the potential energy of the system

$$\Pi = U - P\lambda = \frac{1}{2}\int_0^l \frac{M^2 dz}{EJ} - \frac{1}{2}P\int_0^{l_0} y'^2 dz \tag{2.1}$$

where l_0 is the coordinate of the point of application of the force, and determine the extraneous unknown quantities X_i from the condition of stationarity of Π for conservative external forces

$$\partial\Pi / \partial X_1 = 0, \quad \partial\Pi / \partial X_2 = 0, \dots \tag{2.2}$$

4. Equating expression (2.1) to zero, we arrive at Eq. (1.5), from which we determine the critical force.

As a result, a value of the force $P = P_*$ is established for which the equilibrium form of the bending force can exist. This will be the approximate value of the critical force.

To illustrate the proposed method, we will consider two examples of statically indeterminate problems.

Example 1. It is required to determine the critical force for a rod loaded as shown in Fig. 2, with EJ = const.

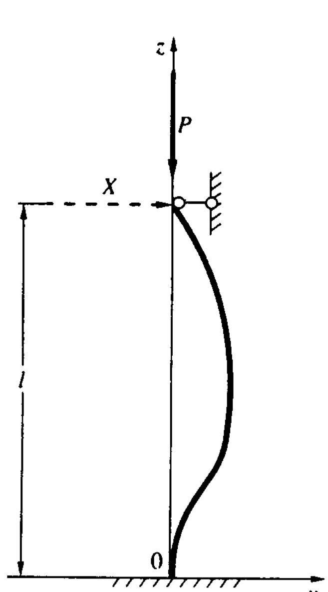

Fig. 2

We will specify approximately the deflection function in the form of a fourth-degree polynomial

$$y = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0$$

the coefficients of which will be determined from the four boundary conditions

$$z = 0: \quad y = y' = 0; \quad z = l: \quad y = y'' = 0$$

The deflection function that satisfies these conditions has the form

$$y = Af(z), \quad f(z) = z^4 - \frac{5}{2} l z^3 + \frac{3}{2} l^2 z^2$$

The bending moment is determined by the equation of equilibrium of the intercepted part of the rod in the strained state

$$M = X(l - z) - Py = X(l - z) - PAf(z)$$

where X is the unknown reaction at the upper support.

The strain potential energy U, the displacement of the point of application of the force, and the potential energy of the system Π have the form

$$U = \frac{X^2 l^3}{6EJ} - \frac{XPAl^6}{30EJ} + \frac{19P^2A^2l^9}{5040EJ}$$

$$\lambda = \frac{1}{2}\int_0^l y'^2 dz = \frac{3}{70} A^2 l^7$$

$$\Pi = U - P\lambda$$

Reaction X is determined by the condition $\partial\Pi/\partial X = 0$, whence we obtain

$$X = PAl^3/10$$

The strain potential energy corresponding to the value of X obtained is equal to

$$U = 159P^2A^2l^9/(75600EJ)$$

From the condition $\Pi = 0$ we find the critical force

$$P_* = 20.4EJ/l^2$$

which differs from the accurate value [2] by 1%.

The solution of the same problem by means of formulae (1.1) and (1.2) gives

$$P_* = 21EJ/l^2$$

which differs from the accurate solution by 4%.

Thus, the energy method based on formula (1.3) leads to a more accurate result than the approximate solution using formula (1.2).

Example 2. It is required to determine the critical force for a rod loaded as shown in Fig. 3, with EJ = const.

The boundary conditions and the corresponding deflection function have the form

$$z = 0: \quad y = y' = 0; \quad z = l/2: \quad y = 0; \quad z = l: \quad y = y'' = 0,$$

$$y = Af(x), \quad f(x) = x^5 - \frac{11}{4}x^4 + \frac{19}{8}x^3 - \frac{5}{8}x^2$$

$$x = z/l$$

The displacement of the point of application of the force

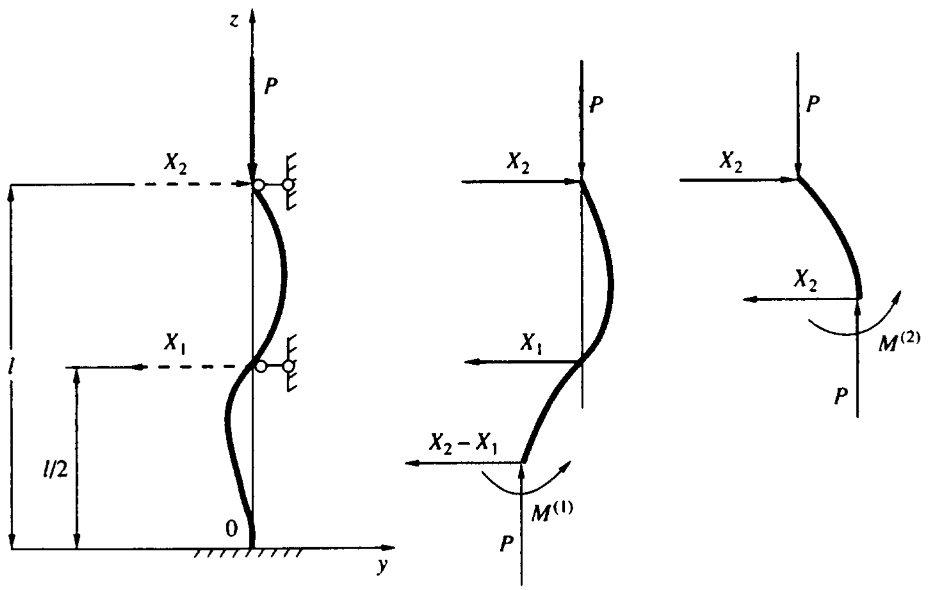

Fig. 3

$$\lambda = 23A^2/(10080l)$$

The bending moments are determined by the equations of equilibrium of the intercepted parts of the rod in the strained state (the right-hand part of Fig. 3)

$$M^{(1)} = -Py - X_1(l/2 - z) + X_2(l - z)$$

$$M^{(2)} = -Py + X_2(l - z)$$

where X_1 and X_2 are extraneous unknown quantities.

The strain potential energy

$$U = \frac{1}{2EJ}\left[\int_0^{l/2} M^{(1)^2} dz + \int_{l/2}^{l} M^{(2)^2} dz\right] \tag{2.3}$$

The extraneous unknown quantities are determined by conditions (2.2), which are obviously equivalent to the equations

$$\partial U / \partial X_1 = 0, \quad \partial U / \partial X_2 = 0$$

We differentiate function (2.3) with respect to X_1 and X_2. We thereby obtain a system of two equations in a form similar to the canonical system of the method of forces with transverse bending [2],

$$\delta_{i1}X_1 + \delta_{i2}X_2 + \Delta_{i0} = 0, \quad i = 1,2 \tag{2.4}$$

$$\delta_{11} = \frac{1}{EJ}\int_0^{l/2}\left(\frac{l}{2} - z\right)^2 dz, \quad \delta_{12} = \delta_{21} = -\frac{1}{EJ}\int_0^{l/2}(l - z)\left(\frac{l}{2} - z\right)dz$$

$$\delta_{22} = \frac{1}{EJ}\int_0^{l}(l - z)^2 dz$$

$$\Delta_{10} = \frac{P}{EJ}\int_0^{l/2} y\left(\frac{l}{2} - z\right)dz, \quad \Delta_{20} = -\frac{P}{EJ}\int_0^{l} y(l - z)dz$$

To determine the coefficients δ_{ij} one can use Vereshchagin's rule, which replaces the evaluation of integrals by multiplication of the corresponding curves (Fig. 4), which considerably reduces the volume of calculations. To the left of the diagram in Fig. 4 we show a graph of the function y which, in the

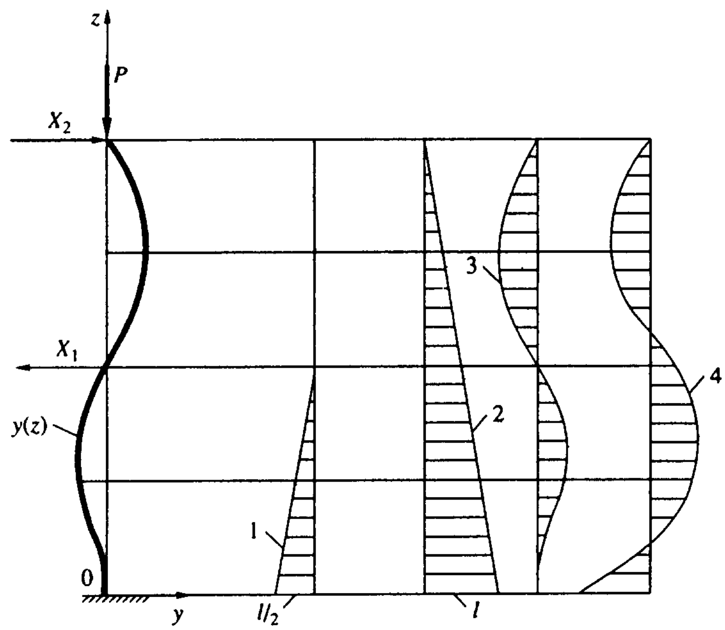

Fig. 4

characteristic section $z = l/4$, has the value $y = 3A/256$, and in the characteristic section $z = 3l/4$ the value $y = 9A/512$. Curves 1 and 2 correspond to unit forces $X_1 = 1$ and $X_2 = 1$, curve 3 is a graph of the bending moments as a function of the prescribed system of forces ($-Py$), and curve 4 is a curve of the bending moments in the system after expansion of the static indeterminacy. In particular, in the section at the fixing, the moment amounts to $1528PA/125440$.

To determine the coefficients Δ_{10} and Δ_{20}, analytical integration is necessary, since the function $y(z)$ may be arbitrary.

As a result, from system of equations (2.4) we obtain

$$X_1 = 9PA/(196l), \quad X_2 = 169PA/(15680l).$$

The strain potential energy is calculated by means of formula (2.3):

$$U = \frac{P^2}{2EJ}\int_0^l y^2 dz + \frac{1}{2}\delta_{11}X_1^2 + \delta_{12}X_1X_2 + \frac{1}{2}\delta_{22}X_2^2 + \Delta_{10}X_1 + \Delta_{20}X_2 = 4.34 \cdot 10^{-5}\frac{P^2A^2l}{EJ}$$

The critical force is determined from the condition $\Pi = 0$, from which it follows that

$$P = P_* = 52.5EJ/l^2$$

We will compare the accuracy of the approximate solution obtained with the critical force determined by means of formulae (1.1) and (1.2):

$$P_* = \xi\frac{EJ}{l^2}, \quad \xi = \begin{cases} 56 & \text{by formula (1.1)} \\ 51.12 & \text{for accurate solution [5]} \end{cases}$$

Thus, in the example considered the error of using the calculation of U by means of formula (1.2) amounts to 9.5%, while calculation of U by means of formula (1.3) leads to an error of 2.7%.

An approximate solution of other problems of the stability of compressed rods can also be obtained in a similar way. For example, for the rod shown in Fig. 5, with $l_0 = 0.3l$ we have

$$P_* = \xi\frac{EJ}{l^2}, \quad \xi = \begin{cases} 81.5 & \text{by formulae (1.1) and (1.2)} \\ 74.4 & \text{by formula (1.3)} \\ 64 & \text{for the accurate solution} \end{cases}$$

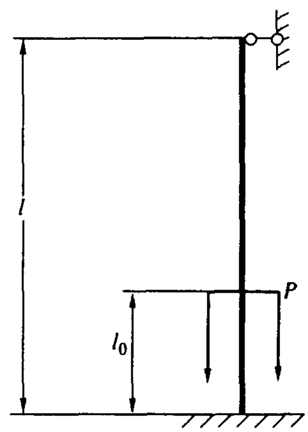

Fig. 5

Thus, the calculation of the potential energy by means of formula (1.3) leads, in a number of cases, to better agreement with the accurate solution than calculation of U by means of formula (1.2).

3. THE ENERGY METHOD IN VIBRATION THEORY

The method described for determining the extraneous unknown quantities in statically indeterminate systems can also be used in problems of determining the natural frequencies of vibrations of rods. The Grammel method [6], widely used in vibration theory, was used in [7] to solve the statically determinate problem of calculating the frequency of transverse vibrations of a cantilever beam. Taking an elementary example, we will show that it is possible to extend Grammel's method to statically indeterminate systems.

Consider the longitudinal vibrations of a rod of constant cross-section, fastened at its ends [7]. We will assume an amplitude function in the form

$$u = \sin(\pi z/l)$$

which corresponds to an accurate form of the natural vibrations of the first frequency.

Then, the maximum kinetic energy of motion and the maximum strain potential energy are calculated by means of the formulae

$$T_{\max} = \frac{1}{2}p^2\int_0^l u^2 dz = \frac{1}{4}p^2\rho Fl$$

$$U_0 = \frac{1}{2EF}\int_0^l \left[X - p^2\rho F\frac{l}{\pi}\left(1-\cos\frac{\pi z}{l}\right)\right]^2 dz = \frac{l}{2\pi EF}\left(\pi X^2 - 2Xp^2\rho Fl + \frac{3}{2}p^4\rho^2F^2\frac{l^2}{\pi}\right)$$

where p is the frequency of natural vibrations, and X is the unknown reaction at the support.

To determine the reaction in a statically indeterminate system, we use the equation

$$\partial U_0/\partial X = 0$$

essentially equivalent to Hamilton's variational principle. We then obtain

$$X = p^2\rho Fl/\pi, \quad U_0 = p^4\rho^2F^2l^3/(4\pi^2 EF)$$

and the equation $T_{\max} = U_0$ leads, as expected, to an accurate solution for the first frequency

$$p = \pi[E/(\rho l^2)]^{1/2}$$

Another example is provided by the transverse vibrations of rod rigidly clamped at its ends. In this case, for a rod of length l, the accurate solution of the first natural frequency [7] is

$$p_1 = 22.4\sqrt{EJ/(m_0 l^4)} \tag{3.1}$$

Approximate solution by Rayleigh's formula

$$p^2 = \int_0^l EJ\left(\frac{d^2u}{dz^2}\right)^2 dz\left(\int_0^l m_0 u^2 dz\right)^{-1}$$

for the function $u = 1 - \cos(2\pi z/l)$ leads to a value differing from (3.1) in a factor of 22.8 instead of 22.4.

Using Grammel's method, for the half-rod we take $u = 1 + \cos(2\pi z/l)$. We than obtain the following expression for the maximum kinetic energy of motion and the intensity of the inertia forces

$$T_{\max} = 3p^2 m_0 l/8, \quad q = p^2 m_0(1+\cos x), \quad x = 2\pi z/l$$

The transverse force in the section

$$Q = \int_0^z q dz = \frac{p^2 m_0 l}{2\pi}(x + \sin x)$$

The bending moment

$$M = -X + \int_0^z Q dz = -X + p^2 m_0\left(\frac{l}{2\pi}\right)^2\left(\frac{x^2}{2} - \cos x + 1\right)$$

where X is the unknown bending moment in the middle section.

Calculation of the strain potential energy by means of the formula

$$U_0 = \frac{1}{2}\int_0^l \frac{M^2 dz}{EJ}$$

and the determination of X from the condition $\partial U_0/\partial X_0$ give

$$X = p^2 m_0\left(\frac{l}{2\pi}\right)^2\left(\frac{\pi^2}{6} + 1\right)$$

$$U_0 = 14.65\frac{p^4 m_0^2 l}{4\pi EJ}\left(\frac{l}{2\pi}\right)^4$$

Setting $T_{\max} = U_0$, we obtain an accurate value of the first natural frequency.

Thus, the use of the method described to determine the extraneous unknown quantities in problems of the stability and vibrations of rods improves the accuracy of the approximate solutions obtained.

REFERENCES

1. TIMOSHENKO, S. P. and GERE, T. M., *Theory of Elastic Stability*. McGraw-Hill, New York, 1959.
2. FEODOS'YEV, V. I., *The Strength of Materials*. Nauka, Moscow, 1986.
3. VOL'MIR, A. S., *The Stability of Elastic Systems*. Fizmatgiz, Moscow, 1963.
4. ALFUTOV, N. A., *Principles of the Stability Analysis of Elastic Systems*. Maskinostroyeniye, Moscow, 1978.
5. PONOMAREV, S. D. (Ed.), *Strength Calculations in Machine Construction*, Vol. 3. Mashgiz, Moscow, 1959.
6. BIEZENO, C. B. and GRAMMEL, R., *Technische Dynamik*. Springer, Berlin, 1939.
7. BIDERMAN, V. L., *Applied Theory of Mechanical Vibrations*. Vysshaya Shkola, Moscow, 1972.

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